The Andante Regime of Scalar Field Dynamics

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Abstract

The andante regime of scalar field dynamics in the chaotic inflationary Universe is defined as the epoch when the field is rolling moderately slowly down its interaction potential, but at such a rate that first-order corrections to the slow-roll approximation become important. These conditions should apply towards the end of inflation as the field approaches the global minimum of the potential. Solutions to the Einstein-scalar field equations for the class of power law potentials $V(\phi) \propto \phi^{2n}$ are found in this regime in terms of the inverse error function.

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1 Introduction

A study of the evolution of self-interacting scalar fields in the early Universe is important for a number of reasons. Firstly, in addition to offering a possible resolution to some of the fundamental problems of the hot big bang model, the inflationary scenario provides a causal mechanism for generating adiabatic density perturbations [1]. These may produce anisotropies in the cosmic microwave background and act as a seed for galaxy formation via gravitational instability. During inflation the Universe is dominated by the potential energy $V(\phi)$ associated with the self-interactions of a quantum scalar field ϕ. If the field is initially displaced from the global minimum of the potential and the potential is sufficiently flat, the scalar field will evolve very slowly towards the true vacuum state. The potential therefore behaves as an effective cosmological constant and introduces a negative pressure into the Universe that drives the accelerated expansion. Secondly, many natural extensions to General Relativity may be expressed in terms of Einstein gravity minimally coupled to a self-interacting scalar field after a suitable conformal transformation on the metric tensor. Two classes of theory that have received much attention in recent years are scalar-tensor theories of gravity and higher-order theories, where the gravitational lagrangian is an analytic function of the Ricci scalar R.

However, only a limited number of exact solutions to the isotropic Einstein-scalar field equations have been found to date. These have recently been classified by Barrow in terms of the potential $V = V_0 \phi^N \exp(A\phi^M)$, where $\{V_0, A, M, N\}$ are constants [2]. Special cases include $V =$ constant corresponding to exponential expansion, power-law inflation from an exponential potential $(N = 0, M = 1)$ [3, 5, 6, 7] and intermediate inflation from a combination of power-law potentials $(N < 0, M = 0)$ [4, 8]. Solutions corresponding to potentials leading to hybrid inflation [9] have been presented in Refs. [10, 11] and exact solutions have also been found for hyperbolic and trigonometric potentials [12, 13].

Consequently, it is common practice to invoke the *slow-roll* approximation. This assumes that the kinetic energy of the scalar field is negligible relative to its potential energy and that the dominant term in the scalar field equation is due to friction arising from the expansion of the Universe. The system is therefore reduced to a set of coupled, first-order differential equations. The slow-roll approximation is usually valid during the initial stages of the inflationary expansion, but as the field rolls towards the minimum, the approximation inevitably breaks down at some point. Moreover, inflation arises whenever the strong energy condition is violated and this does not necessarily require the slow-roll approximation to be valid.

Therefore, it is important to develop analytical techniques that allow deviations from the slow-roll regime to be accounted for and in this paper we discuss how this may be achieved. After summarizing the general features of scalar field dynamics in Sec. II, we illustrate how corrections to the slow-roll approximation may be included in Sec. III and we derive the corresponding field equations. In Sec. IV these equations are solved in parametric form for the class of power law potentials $V(\phi) \propto \phi^{2n}$ in terms of the inverse error function. The rapid increase in the scale factor during inflation is naturally explained by using the properties of this function.

2 Scalar Field Dynamics

Considerable progress in determining scalar field dynamics has been made recently by treating the scalar field as the dynamical variable, since this reduces the field equations to a set of first-order, non-linear differential equations [4, 5, 6, 13, 14, 15]. The four-dimensional action for Einstein gravity minimally coupled to a self-interacting scalar field is

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right], \tag{1}$$

where $g \equiv \det g_{\mu\nu}$, $\kappa^2 \equiv 8\pi m_P^{-2}$ and m_P is the Planck mass. We choose units such that $c = \hbar = 1$. If the Universe is spatially isotropic, closed and flat with a world-interval $ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[dx^2 + dy^2 + dz^2]$ and lapse function $N(t)$, the Arnowitt-Deser-Misner (ADM) action is

$$S = \int dtUNe^{3\alpha}\left[-\frac{3}{\kappa^2}\frac{\dot{\alpha}^2}{N^2} + \frac{1}{2}\frac{\dot{\phi}^2}{N^2} - V(\phi)\right], \tag{2}$$

where $U \equiv \int d^3\mathbf{x}$ is the comoving volume of the Universe and a dot denotes differentiation with respect to cosmic time t [16]. The classical Hamiltonian constraint $\mathcal{H} = 0$ implies that the action satisfies the Hamilton-Jacobi equation

$$-\frac{\kappa^2}{6}\left(\frac{\partial S}{\partial\alpha}\right)^2 + \left(\frac{\partial S}{\partial\phi}\right)^2 + 2U^2e^{6\alpha}V(\phi) = 0. \tag{3}$$

The classical dynamics of this model is determined by the real, separable solution

$$S = -\frac{2}{\kappa^2}Ue^{3\alpha}H(\phi), \tag{4}$$

where $H(\phi)$ satisfies the differential equation [4, 5, 6]

$$\left(\frac{dH}{d\phi}\right)^2 - \frac{3\kappa^2}{2}H^2(\phi) = -\frac{1}{2}\kappa^4V(\phi). \tag{5}$$

This equation is equivalent to the 00-component of the Einstein field equations and therefore represents the Friedmann equation. In the conformal gauge $N = 1$ the momenta conjugate to α and ϕ are $p_\alpha = \partial S/\partial\alpha = -6\kappa^{-2}Ue^{3\alpha}\dot{\alpha}$ and $p_\phi = \partial S/\partial\phi = Ue^{3\alpha}\dot{\phi}$, respectively. Substitution of ansatz (4) into these expressions implies that

$$H(\phi) = \dot{\alpha}, \qquad -\frac{2}{\kappa^2}\frac{dH}{d\phi} = \dot{\phi} \tag{6}$$

and it follows that $H(\phi)$ is the Hubble expansion parameter expressed as a function of the scalar field ϕ. The general solution to the field equations (5) and (6) may be expressed in terms of quadrature with respect to this function:

$$\alpha[\phi(t)] = \alpha_i - \frac{\kappa^2}{2}\int_{\phi_i}^{\phi} d\phi' H(\phi', p)\left(\frac{\partial H(\phi', p)}{\partial\phi'}\right)^{-1}, \tag{7}$$

where (α_i, ϕ_i) are arbitrary constants and p is a parameter associated with each solution $H(\phi, p)$ of Eq. (5) [5, 6].

In principle the general path of the Universe in the minisuperspace (α, ϕ) is uniquely determined once the functional form of $H(\phi)$ is specified and this suggests that it should be be viewed as the fundamental quantity in the analysis. Unfortunately, however, it is rather difficult to find exact solutions to Eq. (5) that go beyond the slow-roll approximation. In view of this it has been suggested that one could begin the analysis by specifying $H(\phi)$ [11, 17]. An alternative approach is to generate exact solutions by viewing the expansion parameter as the effective time coordinate [18]. The drawback with these approaches, however, is that the exact solutions do not necessarily correspond to realistic potentials. Moreover, since it is the potential, and not $H(\phi)$, that is specified by the particle physics of the model, one should aim to solve the model by specifying the potential. In the next section we show how an approximation to Eq. (5) can be made when the slow-roll approximation is not valid.

3 Beyond Slow-Roll: The Andante Regime

It proves convenient to introduce the rescaled quantities [4]

$$u \equiv \left(\frac{\kappa^2}{3} V \right)^{1/2}, \qquad x \equiv \left(\frac{3}{2} \right)^{1/2} \kappa\phi \tag{8}$$

since Eq. (5) then simplifies to

$$H^2 = (H')^2 + u^2. \tag{9}$$

Deviations from the slow-roll regime have been analyzed previously by Salopek and Bond [5]. In order to derive a more accurate solution they substituted the zeroth-order approximation $H_{(0)} = u$ into the right-hand side of Eq. (9):

$$H_{(1)}^2 = H_{(0)}^2 \left[1 + \left(\frac{d \ln H_{(0)}}{dx} \right)^2 \right] \tag{10}$$

and further accuracy is achieved by including higher-order terms.

However, deviations from the slow-roll approximation may be studied more quantitatively in terms of the dimensionless parameters

$$\begin{aligned} \epsilon &\equiv \frac{3\dot{x}^2}{\dot{x}^2 + 9u^2} = 3 \left(\frac{H'}{H} \right)^2 \\ \eta &\equiv -\frac{\ddot{x}}{H\dot{x}} = 3\frac{H''}{H} = \epsilon - \frac{1}{2} \left(\frac{3}{\epsilon} \right)^{1/2} \epsilon' \end{aligned} \tag{11}$$

and

$$\xi \equiv 3\frac{H'''}{H'} = \eta - \left(\frac{3}{\epsilon} \right)^{1/2} \eta', \tag{12}$$

where a prime denotes differentiation with respect to the dimensionless field x [19]. For convenience we consider $\dot{x} > 0$ in this work, which implies that $\sqrt{\epsilon} = -\sqrt{3}H'/H$. Modulo

a constant of proportionality, ϵ is a measure of the field's kinetic energy relative to its total energy, whilst η measures the field's acceleration relative to the amount of friction acting on it due to the expansion of the Universe. We may therefore refer to these quantities as the *energy* and *friction* parameters, respectively [12]. The slow-roll approximation to scalar field dynamics corresponds to $\{|\epsilon|, |\eta|, |\xi|\} \ll 1$ and in this regime the energy and friction parameters reduce to the slow-roll parameters introduced in Ref. [20]. It is straightforward to show that inflation occurs if $\epsilon < 1$ and the end of inflation can be defined precisely by the condition $\epsilon = 1$ [5, 6, 17].

Eq. (9) can be written in a very illuminating form by introducing the parameter

$$v \equiv \sqrt{\frac{3}{\epsilon}}, \qquad v' = \frac{\eta}{\epsilon} - 1. \tag{13}$$

It follows that

$$\frac{1}{v^2} = \frac{\epsilon}{3} = 1 - \frac{u^2}{H^2} \tag{14}$$

and differentiation of this equation with respect to x implies that

$$\frac{u'}{u} = -\frac{1}{v} + \frac{v'}{v(v^2-1)}. \tag{15}$$

A comparison of Eqs. (13) and (15) implies that the 'steepness' of the potential, as defined by Turner [21], may now be expressed directly in terms of the energy and friction parameters:

$$\frac{u'}{u} = -\sqrt{\frac{\epsilon}{3}} \left[\frac{1-\eta/3}{1-\epsilon/3}\right]. \tag{16}$$

The rate of change of steepness in the potential can also be written with these parameters and has the form

$$\frac{1}{\kappa^2 V}\frac{d^2V}{d\phi^2} = 3\left[\left(\frac{u'}{u}\right)^2 + \frac{u''}{u}\right] = \frac{\eta[1-\eta/3] + \epsilon[1-\xi/3]}{3[1-\epsilon/3]}. \tag{17}$$

Hence, the first slow-roll condition corresponds to $u'/u \approx -\sqrt{\epsilon/3} \ll 1$, i.e. $H(x) \approx u(x)$.

We may consider first-order departures from the slow-roll regime by expanding the terms in the square bracket of Eq. (16) to first-order in ϵ and η. With the help of Eq. (13) we find

$$\frac{u'}{u} \approx -\frac{1}{v} + \frac{v'}{v^3} + \mathcal{O}\left(\frac{v'}{v^5}\right) \tag{18}$$

and the dymanics of the scalar field in this regime of parameter space is then determined by this equation. Eq. (18) may also be derived by expanding the right-hand side of Eq. (15) as a geometric progression for $v > 1$:

$$\frac{u'}{u} = -\frac{1}{v} + \frac{v'}{v^3}\left(1 + \frac{1}{v^2} + \frac{1}{v^4} + \ldots\right). \tag{19}$$

It is seen, therefore, that accounting for first-order deviations from the slow-roll regime is equivalent to including the first term in this series.

For a specific choice of potential the solution to Eq. (18) will determine $v(x)$ and the functional form for $H(x)$ then follows directly from Eq. (14). Eq. (18) describes the dynamics of a scalar field that is evolving moderately slowly down its self-interaction potential but at a sufficiently fast rate that the usual slow-roll approximation is no longer valid. Consequently we refer to this region as the moderately slow, or *andante*, regime of scalar field dynamics[2]. The slow-roll limit corresponds to the regime where the energy and friction parameters are negligible with respect to unity and the andante regime applies when these parameters are small but finite. Expressions for the amplitudes of the scalar and tensor fluctuations when these latter conditions apply have been derived previously by Stewart and Lyth [22].

An alternative form of Eq. (18) may be derived by viewing the volume of the Universe as the effective dynamical variable. Since the volume is a monotonically increasing function, it is a suitable choice for the 'time' coordinate. It follows from the general solution (7) that the number of e-foldings of expansion that occur as the scalar field rolls from some initial value x_i to a value x is given by

$$\begin{aligned}\ln a \propto N(x) = \int_{t_i}^{t} dt' H(t') &= -\frac{1}{3}\int_{x_i}^{x} dx H(x') \left(\frac{dH(x')}{dx'}\right)^{-1} \\ &= \frac{1}{3}\int_{x_i}^{x} dx' v(x'). \end{aligned} \tag{20}$$

Hence, the ratio $z/z_i = (a/a_i)^3 = \exp(3N)$ represents the fractional increase in the volume of the universe that occurs when the field rolls from x_i to x and it follows immediately from the definition (13) that

$$\frac{dx}{dz} = \frac{1}{zv}. \tag{21}$$

If we now define a new function $m(z)$:

$$m(z) \equiv zx(z), \tag{22}$$

it can be shown, after differentiating with respect to z and substituting in Eq. (18), that

$$\frac{dm}{dz} = x + \frac{1}{v} \tag{23}$$

and

$$m\frac{d^2m}{dz^2} = -x\frac{u'}{u}. \tag{24}$$

We shall show in the following section that Eq. (24) is exactly solvable when the potential is given by $V(\phi) \propto \phi^{2n}$.

[2] Similarly the slow-roll regime could be referred to as the 'largo' regime, whilst the epoch after inflation that arises when the field undergoes rapid oscillations about the minimum would represent the 'allegro' regime.

4 Power Law Potentials

In this section we consider the class of power law potentials

$$V(\phi) = \lambda \phi^{2n}, \qquad n = \text{constant}. \tag{25}$$

where λ is the coupling constant. In the chaotic inflationary scenario, the andante regime should apply during the final stages of the inflationary expansion when the scalar field approaches the global minimum of its potential. In this region it is useful to consider the general inflationary potential as a truncated Taylor series expanded about this minimum. It is therefore a good approximation to assume the potential has the power law form (25) with positive-definite n. However, the class of potentials with $n < 0$ is also interesting and potentials of this form may be important if the Universe contains a nonvanishing vacuum energy at the present epoch [23]. Moreover, they arise in generalized scalar-tensor theories when the Brans-Dicke parameter is viewed as a truncated Taylor series in the dilaton field [24].

When the potential is a power law, Eq. (24) admits the first integral

$$\frac{dm}{dz} = \pm \left(c - 2n \ln m\right)^{1/2}, \tag{26}$$

where c is an arbitrary constant. For consistency we require $m > 0$ in Eq. (26), and since we assumed $\dot{x} > 0$, this implies that both x and z must be negative if $n > 0$ and they must be positive if $n < 0$. However, $|z|$ provides a measure of the amount of inflation that occurs and it varies much more rapidly than the scalar field x. Thus, the evolution of the function m is dominated by z, which implies that m increases as $|z|$ increases. Consequently, the negative square root should be chosen when $n > 0$ and the positive square root corresponds $n < 0$.

Eq. (26) is solved exactly in terms of the error function:

$$z = b + \gamma \text{erf}\left[\mp\left(\frac{c}{2n} - \ln(zx)\right)^{1/2}\right], \tag{27}$$

where b is the second integration constant and

$$\gamma \equiv \left(\frac{\pi}{2n}\right)^{1/2} e^{c/(2n)}. \tag{28}$$

This solution may be derived from the identity $d[\text{erf}(y)]/dy = (2/\sqrt{\pi})\exp(-y^2)$ [25].

Eq. (27) can be inverted to yield

$$x(z) = e^{c/(2n)} \frac{1}{z} \exp\left[-\left(\text{Ierf}\left(\frac{z-b}{\gamma}\right)\right)^2\right], \tag{29}$$

where $\text{Ierf}(y)$ is the inverse error function and the expression for $v(z)$ follows by substituting Eqs. (26) and (29) into Eq. (23):

$$\frac{1}{v} = -x - \sqrt{2n}\,\text{Ierf}\left(\frac{z-b}{\gamma}\right). \tag{30}$$

Finally we find $z(x)$ by substituting Eq. (30) into Eq. (29):

$$z(x) = e^{c/(2n)} \frac{1}{x} \exp\left[-\frac{1}{2n} \left(x + \frac{1}{v(x)} \right)^2 \right]. \tag{31}$$

These solutions exhibit the correct qualitative behaviour that one would expect for a scalar field rolling down a polynomial potential. Figures 1a and 1b illustrate the behaviour of the functions $x(z)$ and $v^{-1}(z)$ for $n = 1$, $c = 0$ and $b = -\gamma$ and similar results are found for others choices of these constants. A relatively small change in the value of the scalar field in the range $-30 \leq x \leq -10$ results in a huge change in the value of z by a factor $\sim 10^{180}$. The origin of this behaviour is traced to the properties of the inverse error function. $\mathrm{Ierf}(y)$ is an odd function of y and is undefined for $|y| > 1$. For positive arguments it is a monotonically increasing function with $\mathrm{Ierf}(0) = 0$ and $\mathrm{Ierf}(1) = \infty$. However, the slightest deviation of the argument from unity results in a sharp decrease in the value of the inverse error function. For example, $\mathrm{Ierf}(1 - 10^{-1000}) = 47.9$ and $\mathrm{Ierf}(1 - 10^{-100}) = 15.1$. This implies that $\mathrm{Ierf}(y)$ is a relatively slowly varying function in the range $0 \leq y \leq 1 - 10^{-10^3}$ and consequently z may change by a factor 10^{10^3} without there being a significant change in the value of the scalar field. This feature leads to a rapid growth in the scale factor for a very small change in the value of the scalar field.

In the limit $|x| \gg v^{-1}$ the functional form of the solution (31) implies that the scale factor may be expressed directly in terms of the scalar field as

$$a(x) \propto \left(\frac{1}{x} \right)^{1/3} e^{-x^2/(6n)}. \tag{32}$$

From this expression it follows that Eq. (20) may be employed to evaluate $H(x)$:

$$H^2 = u^2 \left[1 + \frac{n}{x^2} \right]^n, \tag{33}$$

and if one expands the right-hand side to first-order, the result is equivalent to Eq. (10). Hence, the Salopek-Bond approximation [5] is recovered from the parametric solution (29)-(30) in the limit that v diverges.

Similar arguments apply when $n < 0$. Considering negative values of n is equivalent to Wick rotating γ to the imaginary axis, i.e. $\gamma \to i\gamma$. The error function with imaginary argument is related to the imaginary error function, $\mathrm{erf}(iy) \equiv i\mathrm{erfi}(y)$, and it follows that solutions (29) and (30) take the form

$$x(z) = e^{c/(2n)} \frac{1}{z} \exp\left[\left(\mathrm{Ierfi}\left(\frac{z - b}{\sqrt{|\gamma^2|}} \right) \right)^2 \right] \tag{34}$$

and

$$\frac{1}{v} = -x + \sqrt{2|n|}\,\mathrm{Ierfi}\left(\frac{z - b}{\sqrt{|\gamma^2|}} \right), \tag{35}$$

respectively. Figures 2a and 2b illustrate the evolution of $x(z)$ and $v^{-1}(z)$. The De Sitter solution is the attractor at infinity for these models and the intermediate inflationary solution is recovered at large x [8]. Solutions (34) and (35) illustrate analytically how these asymptotic solutions are reached.

5 Conclusion

In this paper we have considered deviations from the slow-roll approximation by including the first-order contributions from the kinetic energy and acceleration of the field. This allows a more accurate analysis of scalar field dynamics to be performed. Such improvements are expected to be relevant towards the final stages of inflation when the scalar field is close to the global minimum of its potential. In view of this we considered a class of power law potentials and an exact parametric solution to the field equations was found in terms of the inverse error function. This function exhibits some very interesting properties and it was shown how the rapid growth in the scale factor of the Universe during inflation is naturally explained by employing the properties of this function. Since a large increase in the scale factor for a very small change in the value of the scalar field is a generic feature of the inflationary scenario, we conjecture that the inverse error function may arise in the general solution to the field equations. This possibility is currently under investigation. The solutions presented here should improve our understanding of how the Universe moves out of the inflationary epoch and into the reheating phase. It will be interesting to investigate whether solutions to Eqs. (18) and (24) can be found for other potentials.

Note: After this work was completed we received a preprint by Liddle *et al.* that also discusses the slow-roll approximation though with a different method of analysis [26].

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Figure Captions

Figure 1: (a) A plot of the solution (29) representing the evolution of the scalar field x with respect to $-\log_{10} z$ for the quadratic potential $u(x) \propto x$ $(n = 1)$. The numerical values of the integration constants are specified to be $c = 0$ and $b = -(\pi/2)^{1/2}$. (b) The solution (30) for the same choice of parameters as in Figure 1a.

Figure 2: (a) Illustrating the solution (34) for the decaying power law potential $u \propto x^{-1}$ $(n = -1)$, where $b = c = 0$ and $\gamma = (\pi/2)^{1/2}$. (b) The corresponding solution (35). The solutions approach the intermediate inflationary solution and the De Sitter solution is the attractor at infinity. Hence, there is no exit from inflation for these potentials.